

if $\Delta\delta_1$ and $\Delta\pi$ be the corrections to the parallactic coefficient and solar parallax

$$\begin{aligned}\Delta\pi &= + \frac{1}{14.2} \Delta\delta_1 = \frac{1}{14.2} \left\{ -\frac{3788}{3074} \Delta\mu + \frac{1366}{3074} \Delta\delta_2 \right\} \\ &= - \frac{1}{11.5} \Delta\mu + \frac{1}{32.0} \Delta\delta_2\end{aligned}$$

It should be mentioned that Newcomb in 1876 attributed the apparent increase of the variation to systematic errors of observation varying with the age of the Moon.

On Oscillating Satellites. (Second Paper.)

By H. C. Plummer, M.A.

1. In the earlier paper (*Monthly Notices*, vol. lxiii. p. 436), to which the present is a continuation, periodic orbits which are possible in the neighbourhood of centres of libration have been investigated with a degree of approximation corresponding to the second order of the relative coordinates. The results appear to suggest that a still closer approximation may be made with some advantage, although the discussion relates only to a small part of the restricted problem of three bodies, and has apparently no direct practical bearing. The notation of the former paper will be followed generally, but some changes will be admitted. The three collinear centres of libration alone will be considered here.

2. The general equations of motion of the satellite are taken in the form

$$\ddot{x} - 2\dot{y} = \Omega_{10}, \quad \ddot{y} + 2x = \Omega_{01}$$

where

$$2\Omega = \mu \left(r_1^2 + \frac{2}{r_1} \right) + \nu \left(r_2^2 + \frac{2}{r_2} \right), \quad \Omega_{rs} = \frac{\partial^{r+s}\Omega}{\partial x^r \partial y^s}.$$

Now at the collinear centres of libration $y = 0$ and $\Omega_{10} = \Omega_{01} = 0$; and $\Omega_{rs} = 0$ when s is odd. Hence in relative coordinates the equations of motion are

$$\begin{aligned}\xi - 2\eta &= \Omega_{20}\xi + \frac{1}{2}(\Omega_{30}\xi^2 + \Omega_{12}\eta^2) + \frac{1}{6}(\Omega_{40}\xi^3 + 3\Omega_{22}\xi\eta^2) + \dots \\ \ddot{\eta} + 2\xi &= \Omega_{02}\eta + \Omega_{12}\xi\eta + \frac{1}{6}(3\Omega_{22}\xi^2\eta + \Omega_{04}\eta^3) + \dots\end{aligned} \quad (1)$$

so long as the satellite remains within the region over which the expansions are possible. It is known that periodic solutions of these equations exist, and these can be appropriately expressed by Fourier series of the form

$$\begin{aligned}\xi &= a_0 + a_1 \cos mt + a_2 \cos 2mt + a_3 \cos 3mt + \dots \\ \eta &= b_1 \sin mt + b_2 \sin 2mt + b_3 \sin 3mt + \dots\end{aligned} \quad (2)$$

the orbits being symmetrical with respect to the ξ -axis.

If these expressions for ξ and η be substituted in (1) and powers and products of the circular functions be expressed in terms of circular functions of multiples of mt , there will result an identity between two cosine series and an identity between two sine series. Now it is known that a_1, b_1 are of the first order, and a_0, a_2, b_2 of the second order. The order of the other coefficients then corresponds to their suffixes. For if the series (2) be supposed to terminate at the argument imt , the series obtained by substitution in (1) must terminate at the same argument; since it is easy to see that the argument $i'mt$ ($i' > i$) can only arise in connexion with coefficients of order higher than i . Hence the two identities will give $(2i+1)$ equations involving the $(2i+1)$ coefficients and m ; and these equations, together with the Jacobian integral, will suffice to furnish a solution to the order of approximation considered.

3. The immediate object is to find a solution which shall be valid to the third order. The expansions of (1) and (2) may therefore be supposed to end with the terms actually written down. Then when (2) are substituted in (1) the following seven equations are obtained by neglecting terms of higher order than the third, and equating the coefficients of the sines and cosines of the different multiples of mt :

$$0 = \Omega_{20}a_0 + \frac{1}{4}\Omega_{30}a_1^2 + \frac{1}{4}\Omega_{12}b_1^2 \quad \dots \quad \dots \quad (3)$$

$$0 = (m^2 + \Omega_{20})a_1 + 2mb_1 + L \quad \dots \quad \dots \quad (4)$$

$$0 = (m^2 + \Omega_{02})b_1 + 2ma_1 + M \quad \dots \quad \dots \quad (5)$$

$$0 = (4m^2 + \Omega_{20})a_2 + 4mb_2 + \frac{1}{4}\Omega_{30}a_1^2 - \frac{1}{4}\Omega_{12}b_1^2 \quad \dots \quad (6)$$

$$0 = (4m^2 + \Omega_{02})b_2 + 4ma_2 + \frac{1}{2}\Omega_{12}a_1b_1 \quad \dots \quad (7)$$

$$0 = (9m^2 + \Omega_{20})a_3 + 6mb_3 + A_3 \quad \dots \quad \dots \quad (8)$$

$$0 = (9m^2 + \Omega_{02})b_3 + 6ma_3 + B_3 \quad \dots \quad \dots \quad (9)$$

The quantities L, M, A_3, B_3 are of the third order and represent the expressions

$$L = \frac{1}{2}\Omega_{30}a_1(2a_0 + a_2) + \frac{1}{2}\Omega_{12}b_1b_2 + \frac{1}{8}\Omega_{40}a_1^3 + \frac{1}{8}\Omega_{22}a_1b_1^2 \quad \dots \quad (4')$$

$$M = \frac{1}{2}\Omega_{12}(2a_0b_1 - a_2b_1 + a_1b_2) + \frac{1}{8}\Omega_{22}a_1^2b_1 + \frac{1}{8}\Omega_{04}b_1^3 \quad \dots \quad (5')$$

$$A_3 = \frac{1}{2}\Omega_{30}a_1a_2 - \frac{1}{2}\Omega_{12}b_1b_2 + \frac{1}{24}\Omega_{40}a_1^3 - \frac{1}{8}\Omega_{22}a_1b_1^2 \quad \dots \quad (8')$$

$$B_3 = \frac{1}{2}\Omega_{12}(a_1b_2 + a_2b_1) + \frac{1}{8}\Omega_{22}a_1^2b_1 - \frac{1}{24}\Omega_{04}b_1^3 \quad \dots \quad (9')$$

4. An eighth equation is required, and this can be obtained from the Jacobian integral. Now

$$\dot{\xi} = -ma_1 \sin mt - 2ma_2 \sin 2mt - 3ma_3 \sin 3mt$$

$$\dot{\eta} = mb_1 \cos mt + 2mb_2 \cos 2mt + 3mb_3 \cos 3mt$$

and

$$2\Omega = 2\Omega_0 + \Omega_{20}\xi^2 + \Omega_{02}\eta^2 + \frac{1}{3}\Omega_{30}\xi^3 + \Omega_{12}\xi\eta^2 \\ + \frac{1}{12}\Omega_{40}\xi^4 + \frac{1}{2}\Omega_{22}\xi^2\eta^2 + \frac{1}{12}\Omega_{04}\eta^4$$

In substituting in the equation

$$V^2 = C_o - C + 2(\Omega - \Omega_o)$$

terms of the fourth order are to be retained, and there results an equation involving cosines of the first four multiples of mt . Their coefficients must clearly vanish, since the integral must be satisfied throughout the motion, and the four relations thus obtained must evidently be involved implicitly in the equations (3)–(9). There will thus remain an equation involving the energy constant which takes this form

$$0 = C_o - C + \frac{1}{2}a_1^2(\Omega_{20} - m^2) + \frac{1}{2}b_1^2(\Omega_{02} - m^2) + N \quad \dots \quad (10)$$

where N stands for this expression of the fourth order

$$\begin{aligned} N &= \frac{1}{2}\Omega_{20}(2a_o^2 + a_2^2) + \frac{1}{2}\Omega_{02}b_2^2 \\ &\quad + \frac{1}{4}\Omega_{30}a_1^2(2a_o + a_2) + \frac{1}{4}\Omega_{12}b_1(2a_o b_1 - a_2 b_1 + 2a_1 b_2) \\ &\quad + \frac{1}{32}\Omega_{40}a_1^4 + \frac{1}{16}\Omega_{22}a_1^2 b_1^2 + \frac{1}{32}\Omega_{04}b_1^4 - 2m^2(a_2^2 + b_2^2) \\ &= \frac{1}{4}(a_1 L + b_1 M) - 4m^2(a_2^2 + b_2^2) - 4ma_2 b_2 \quad \dots \quad (10') \end{aligned}$$

The eight equations (3), (4), . . . (10) are sufficient to determine m and the seven coefficients $a_o, a_1, \dots$

5. Since (3), (6), and (7) give a_o, a_2 , and b_2 in terms of a_1, b_1 , and m , the latter quantities are determined by (4), (5), and (10). An approximate solution is obtained by neglecting L, M , and N . This will give

$$\frac{a_1}{b_1} = \frac{-2m}{m^2 + \Omega_{20}} = \frac{m^2 + \Omega_{02}}{-2m} \quad \dots \quad \dots \quad (11)$$

$$0 = (m^2 + \Omega_{20})(m^2 + \Omega_{02}) - 4m^2 \quad \dots \quad \dots \quad (12)$$

$$a_1^2 = (m^2 + \Omega_{02})a^2; \quad b_1^2 = (m^2 + \Omega_{20})a^2 \quad \dots \quad (13)$$

$$C_o - C = (m^4 - \Omega_{20}\Omega_{02})a^2 \quad \dots \quad \dots \quad \dots \quad (14)$$

Then, if L, M , and N be reinstated, it will be necessary to replace a_1 by $a_1 + \delta a_1$, b_1 by $b_1 + \delta b_1$, and m by $m + \delta m$, the three increments being clearly of the second order. Hence

$$0 = (m^2 + \Omega_{20})\delta a_1 + 2m\delta b_1 + 2(ma_1 + b_1)\delta m + L$$

$$0 = 2m\delta a_1 + (m^2 + \Omega_{02})\delta b_1 + 2(mb_1 + a_1)\delta m + M$$

$$0 = (\Omega_{20} - m^2)a_1\delta a_1 + (\Omega_{02} - m^2)b_1\delta b_1 - m(a_1^2 + b_1^2)\delta m + N$$

the quantities L, M, N and the coefficients in these equations being calculated with the approximate values of a_1, b_1 , and m . Forms more convenient for calculation can be deduced by the help of (11), (12), and (13). Thus if

$$\frac{\delta a_1}{a_1} - \frac{\delta b_1}{b_1} = \delta_1, \quad \frac{\delta a_1}{a_1} + \frac{\delta b_1}{b_1} = \delta_2$$

the following form an equivalent set of equations :

$$(m^4 - \Omega_{20}\Omega_{02})\delta m = m^2(b_1^{-1}L + a_1^{-1}M) \quad \dots \quad (15)$$

$$4m\delta_1 = (\Omega_{20} - \Omega_{02})\delta m + b_1^{-1}L - a_1^{-1}M \dots \quad (16)$$

$$(m^4 - \Omega_{20}\Omega_{02})\delta_2 = m^2(\Omega_{20} - \Omega_{02})\delta_1 - m(2m^2 + \Omega_{20} + \Omega_{02})\delta m + a^{-2}N \dots \quad (17)$$

The first of these determines approximately the variation of the period from its limiting value; the two latter give successively δ_1 and δ_2 , whence δa_1 and δb_1 are obtained immediately. The coefficients a_3 and b_3 are found from (8) and (9), and thus the solution to the third order is complete.

6. To the derivatives of the force function given in the former paper must be added those of the fourth order. Their values are given by

$$3\Omega_{40} = -6\Omega_{22} = 8\Omega_{04} = 72(\mu r_1^{-5} + \nu r_2^{-5})$$

A table of numerical values for the case in which $11\mu = 10$, $11\nu = 1$, will now be given, many of the numbers being taken from the earlier paper :

	Family a.	Family b.	Family c.
r_1	0.71751	1.34700	0.94693
r_2	0.28249	0.34700	1.94693
C_0	3.65292	3.53418	3.17322
Ω_{20}	+ 13.9878	+ 6.0955	+ 3.1660
Ω_{02}	- 5.4939	- 1.5478	- 0.0830
$\Omega_{30} = -2\Omega_{12}$	+ 65.076	- 39.279	+ 6.822
$3\Omega_{40} = -6\Omega_{22} = 8\Omega_{04}$	+ 3982.86	+ 1315.79	+ 86.202
m	2.6082	1.6763	1.0706
$m^4 - \Omega_{20}\Omega_{02}$	123.13	17.330	1.5766
$a^{-1} . a_1$	- 1.1439	- 1.1234	- 1.0312
$a^{-1} . b_1$	+ 4.5597	+ 2.9841	+ 2.0766
$a^{-2} . a_0$	+ 10.5685	- 5.1400	+ 0.5887
$a^{-2} . a_2$	- 4.1352	+ 2.6261	- 0.5487
$a^{-2} . b_2$	- 1.9209	+ 1.5800	- 0.2895
$a^{-3} . L$	+ 1234.6	+ 73.93	+ 2.8618
$a^{-3} . M$	+ 3493.8	+ 47.45	+ 1.4737
$a^{-4} . N$	+ 2981.3	- 118.74	- 2.4169
$a^{-3} . A_3$	- 2044.8	- 288.49	- 8.3943
$a^{-3} . B_3$	- 2190.5	- 225.81	- 6.5525
$a^{-2} . \delta m$	- 153.79	- 2.832	- 0.03708
$a^{-3} . \delta a_1$	- 92.47	- 6.227	- 0.3670
$a^{-3} . \delta b_1$	+ 224.79	- 3.652	- 0.5638
$a^{-3} . a_3$	+ 20.19	+ 7.109	+ 0.4530
$a^{-3} . b_3$	+ 33.64	+ 6.500	+ 0.3559

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7. In the former paper the results for orbits of family α were compared with four orbits given by Professor Darwin. The comparison can now be repeated with improved figures. The values of the energy constant C and the parameter a are given by

$$\begin{aligned} 11\ C &= 40.0, & 39.0, & 38.5, & 38.0 \\ \log a &= 8.06428, & 8.47046, & 8.54705, & 8.60356 \end{aligned}$$

The general equations for the family are

$$\begin{aligned} \xi &= +10.5685a^2 - (1.1439a + 92.47a^3) \cos mt - 4.1352a^2 \cos 2mt \\ &\quad + 20.19a^3 \cos 3mt \\ \eta &= (4.5597a + 224.79a^3) \sin mt - 1.9209a^2 \sin 2mt \\ &\quad + 33.64a^3 \sin 3mt \\ m &= 2.6082 - 153.79a^2 \end{aligned}$$

which become in the four cases considered

$$\begin{aligned} (\alpha) \quad \xi &= +.0014 - .0134 \cos mt - .0006 \cos 2mt + .0000 \cos 3mt \\ \eta &= +.0533 \sin mt - .0003 \sin 2mt + .0001 \sin 3mt \\ m &= 2.588 \end{aligned}$$

$$\begin{aligned} (\beta) \quad \xi &= +.0092 - .0362 \cos mt - .0036 \cos 2mt + .0005 \cos 3mt \\ \eta &= +.1405 \sin mt - .0017 \sin 2mt + .0009 \sin 3mt \\ m &= 2.474 \end{aligned}$$

$$\begin{aligned} (\gamma) \quad \xi &= +.0131 - .0443 \cos mt - .0051 \cos 2mt + .0009 \cos 3mt \\ \eta &= +.1705 \sin mt - .0024 \sin 2mt + .0015 \sin 3mt \\ m &= 2.418 \end{aligned}$$

$$\begin{aligned} (\delta) \quad \xi &= +.0170 - .0519 \cos mt - .0067 \cos 2mt + .0013 \cos 3mt \\ \eta &= +.1975 \sin mt - .0031 \sin 2mt + .0022 \sin 3mt \\ m &= 2.361 \end{aligned}$$

The following table gives the points where the orbits cross the axis of x (ξ_0 and ξ_1), the points (ξ' , η') for which $mt = \frac{1}{2}\pi$ and the angle ($2\pi m^{-1}$) through which the system turns in one period of the satellite. In the comparison numbers η' is the ordinate corresponding to the calculated ξ' .

$mt =$	0	π	$\frac{1}{2}\pi$		
	ξ_0	ξ_1	ξ'	η'	$2\pi m^{-1}$
(α)	-.0126	+.0142	+.0020	+.0532	139.1
D.	-.0125	+.0147	"	+.0530	138.3
(β)	-.0301	+.0413	+.0128	+.1396	145.5
D.	-.0304	+.0413	"	+.1395	145.7

Dec. 1903.		Oscillating Satellites.			103
mt=	0	π	$\frac{1}{2}\pi$		$2\pi m^{-1}$
	ξ_0	ξ_1	ξ'	η'	
(γ)	−0354	+0514	+0182	+01690	148°9
D.	−0361	+0523	„	+01691	149°6
(δ)	−0403	+0609	+0237	+01953	152°5
D.	−041	+060	„	+0197	(not determined)

The limiting value of $2\pi m^{-1}$ for an infinitesimal orbit is 138°0. The agreement shown in the above table is much better than in the corresponding table of the previous paper (vol. lxiii. p. 442), and may be considered satisfactory.

8. For the two orbits of family *b* given by Professor Darwin the values of *C* and *a* are given by

$$\begin{aligned} 11\ C &= 38\cdot5, & 38\cdot0 \\ \log a &= 8\cdot64750, & 8\cdot83116 \end{aligned}$$

The general equations for the family are :

$$\begin{aligned} \xi &= -5\cdot1400\ a^2 - (1\cdot1234a + 6\cdot227a^3) \cos mt + 2\cdot6261a^2 \cos 2mt \\ &\quad + 7\cdot109a^3 \cos 3mt \\ \eta &= (2\cdot9841a - 3\cdot652a^3) \sin mt + 1\cdot5800a^2 \sin 2mt \\ &\quad + 6\cdot500a^3 \sin 3mt \\ m &= 1\cdot6763 - 2\cdot832a^2 \end{aligned}$$

which become in the two cases considered

$$\begin{aligned} (\alpha) \quad \xi &= -0101 - 0504 \cos mt + 0052 \cos 2mt + 0006 \cos 3mt \\ \eta &= +01322 \sin mt + 0031 \sin 2mt + 0006 \sin 3mt \\ m &= 1\cdot6707 \\ (\beta) \quad \xi &= -0236 - 0781 \cos mt + 0121 \cos 2mt + 0022 \cos 3mt \\ \eta &= +02012 \sin mt + 0073 \sin 2mt + 0020 \sin 3mt \\ m &= 1\cdot6633 \end{aligned}$$

The comparison is made as in the preceding paragraph

mt=	0	π	$\frac{1}{2}\pi$		$2\pi m^{-1}$
	ξ_0	ξ_1	ξ'	η'	
(α)	−0547	+0449	−0153	+01316	215°5
D.	−0551	+0445	„	+01312	[213°9]
(β)	−0874	+0644	−0357	+01991	216°4
D.	−0875	+0679		+01987	208

The limiting value of $2\pi m^{-1}$ for an infinitesimal orbit is 214°8. There is therefore a discrepancy between the two methods in

regard to the periods of the orbits, Professor Darwin's results indicating a decrease of the period as the orbits increase in size. This is not supported by my figures. The value $[213^{\circ}9]$ really belongs to a neighbouring orbit which is not quite periodic, so that this number is not strictly comparable. On the other hand the orbit (β) contains in the value of ξ_1 a second discrepancy, which is greater than the evidence based on other comparisons would lead us to expect. On the whole it seems quite possible that the determination of this orbit by the method of quadratures has not been so accurate as in other cases.

9. In Professor Darwin's results the range of the energy constant has the limits given by $11C = 40.0$ and $11C = 38.0$. Now orbits of family c only begin to appear after $11C = 35.0$, and therefore orbits of this family for comparison are wanting. But, although the results cannot be submitted to this kind of check, it seems worth while to apply them to drawing a few new orbits of this family. The general equations are

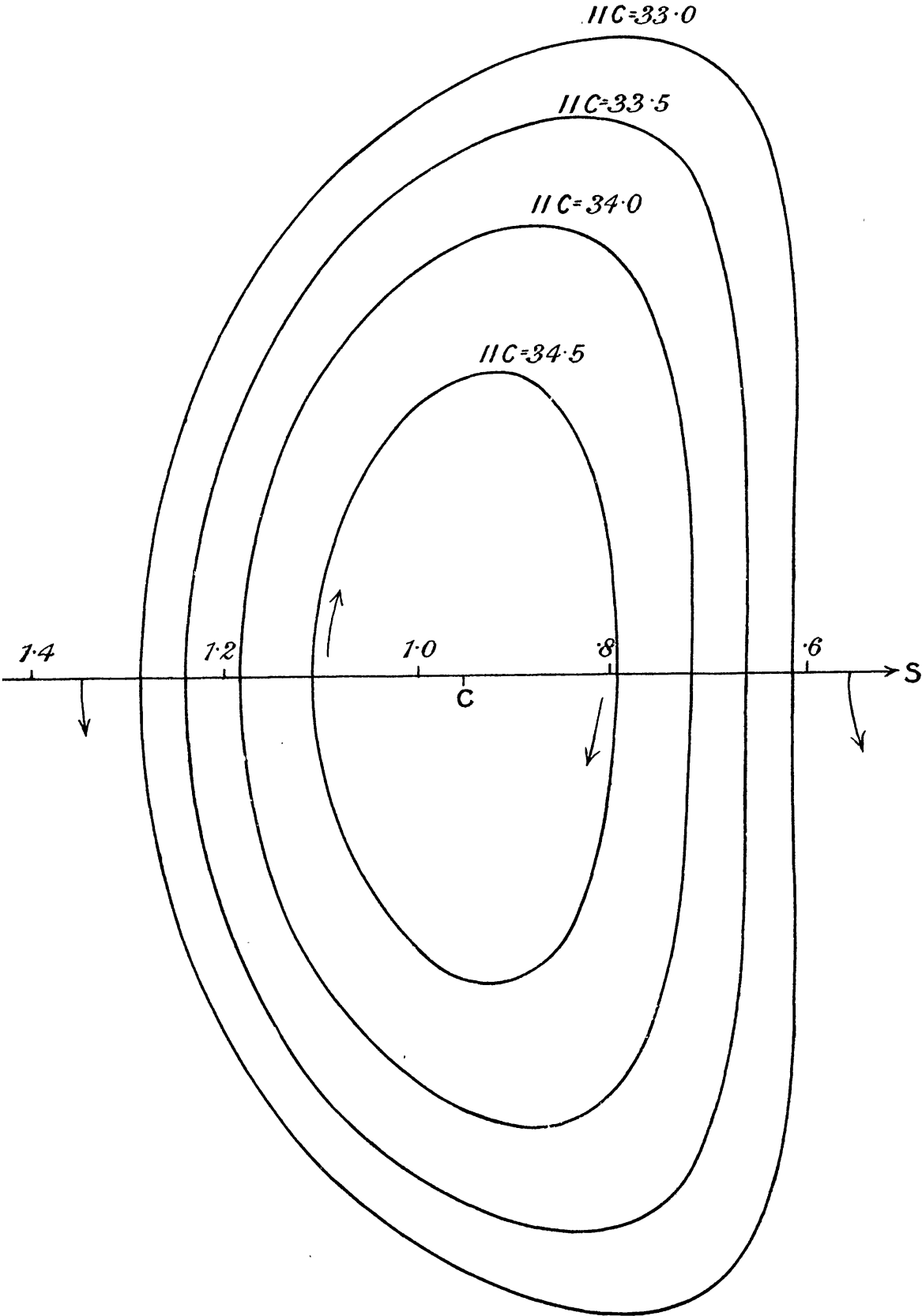
$$\begin{aligned}\xi &= +.5887 a^2 - (.10312 a + .3670 a^3) \cos mt - .5487 a^2 \cos 2mt \\ &\quad + .4530 a^3 \cos 3mt \\ \eta &= (2.0766 a - .5638 a^3) \sin mt - .2895 a^2 \sin 2mt \\ &\quad + .3559 a^3 \sin 3mt \\ m &= 1.0706 - .03708 a^2.\end{aligned}$$

The selected values of C and a are given by

$11C = 34.5$	34.0	33.5	33.0
$\log a = 9.18439$	9.35886	9.45434	9.52043

The equations of the four orbits then become

$$\begin{aligned}(a) \quad \xi &= +.0138 - .1590 \cos mt - .0128 \cos 2mt + .0016 \cos 3mt \\ \eta &= +.3155 \sin mt - .0068 \sin 2mt + .0013 \sin 3mt \\ m &= 1.0697; 2\pi m^{-1} = 336^{\circ}55 \\ (\beta) \quad \xi &= +.0307 - .2400 \cos mt - .0286 \cos 2mt + .0054 \cos 3mt \\ \eta &= +.4678 \sin mt - .0151 \sin 2mt + .0042 \sin 3mt \\ m &= 1.0687; 2\pi m^{-1} = 336^{\circ}85 \\ (\gamma) \quad \xi &= +.0477 - .3020 \cos mt - .0445 \cos 2mt + .0105 \cos 3mt \\ \eta &= +.5782 \sin mt - .0235 \sin 2mt + .0082 \sin 3mt \\ m &= 1.0676; 2\pi m^{-1} = 337^{\circ}20 \\ (\delta) \quad \xi &= +.0647 - .3552 \cos mt - .0603 \cos 2mt + .0165 \cos 3mt \\ \eta &= +.6678 \sin mt - .0318 \sin 2mt + .0130 \sin 3mt \\ m &= 1.0665; 2\pi m^{-1} = 337^{\circ}55\end{aligned}$$



The curves corresponding to these equations are represented in the accompanying plate (Plate 2), in which the unit of length is 5 in., this being the scale on which most of Professor Darwin's orbits are reproduced.

10. A comparison of §§ 7 and 8 with §§ 10 and 11 of the previous paper will show the degree of improvement obtained for families α and b by increasing the order of approximation, and give some idea of the convergence (in a practical and not technical sense) of the series. It has now been found possible to calculate the variation of the period, an element which, with the lower approximation, is supposed constant for each family. In every case the period is found to increase with the scale of the orbits, the variation being most marked in the case of family α and least in the case of family c . As regards the shape and position of the orbits, the general features which characterise the departure from the small elliptic paths surrounding the centres of libration are similar in all cases. The mean position of the satellite tends always in the direction of the nearer body of finite mass, while at the intersection of the orbit with the line of syzygies in the same direction, the curvature first decreases and finally changes sign. These features are shown, though not in a very marked degree, by the orbits of family c drawn in this paper. Where comparison was possible with the results of Professor Darwin's researches it has been made, and in one case among the six orbits available a discrepancy is found which is pointed out in § 8. The method of the present paper is extremely simple and elementary; yet it might be extended without serious difficulty to higher orders of approximation if this were desirable. The next order in particular might easily be added. But it has not seemed likely that anything of value would be gained thereby. A series of orbits of oscillating satellites has been found by M. Burrau (*A.N.* 3251) to be limited by a "trajectory of ejection." The real obstacle in the way of following up to this limit the continuation of the class of solutions considered in this paper is the limited range over which the force function can be expanded in ascending powers of the relative coordinates. Whether this difficulty can be surmounted is a question which must be left for further consideration.

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